Math 172 – Instructional Material

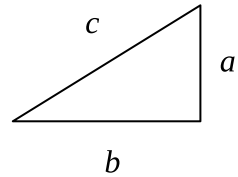
ET Section 7.3 Integration by Trigonometric Substitution

For integrals containing the expression: $\sqrt{a^2-x^2}$, $\sqrt{x^2-a^2}$, *or* $\sqrt{a^2+x^2}$

Recall:

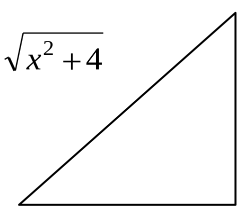
The Pythagorean Theorem states that in any right triangle, the sum of the squares of the legs is equal to the square of the hypotenuse.

$$c^2 = a^2 + b^2$$

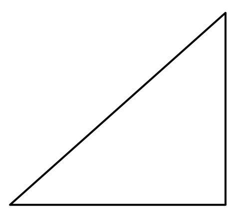


- Notice that this equation can be rewritten in two other ways:
  $c = \sqrt{a^2+b^2}$ or $a = \sqrt{c^2-b^2}$
- Notice that the forms of these radicals match the forms of the integrands we want to work with.
- Our goal is to take the **expressions from the integrals and construct right triangles from them,** so it is important to determine which term is a leg and which is the hypotenuse.
  - If the radicand is the *sum* of two squares, then each of the terms must represent a *leg* of the triangle.
  - If the radicand is the *difference* of two squares, then the *first* term must represent the hypotenuse and the *second* term must represent a leg.
  - The remaining side of the triangle in either case is the radical expression itself.

Exercise 1: Construct two right triangles consistent with the expression $\sqrt{x^2+4}$.



OR



Solution: Since the radicand is the *sum* of two squares, each term is a leg, so the legs are x and 2 , not x^2 and 4, and the hypotenuse is $\sqrt{x^2+4}$.

Exercise 2: Construct two right triangles consistent with the expression $\sqrt{x^2-4}$.

Solution: Since the radicand is the difference of squares, the first term represents the hypotenuse, which is *x*. Label one leg 2 and the other leg is $\sqrt{x^2-4}$.

Exercise 3: Construct two right triangles consistent with the expression $\sqrt{4-x^2}$.

Solution: Since the radicand is the difference of squares, the first term represents the hypotenuse, which is 2. Label one leg *x* and the other leg is $\sqrt{4-x^2}$.

Recall: Right Triangle Trigonometry

If θ is one of the acute angles in a right triangle, then

$$\sin\theta = \frac{\text{side opposite}}{\text{hypotenuse}}, \qquad \cos\theta = \frac{\text{side adjacent}}{\text{hypotenuse}}, \qquad \tan\theta = \frac{\text{side opposite}}{\text{side adjacent}}$$

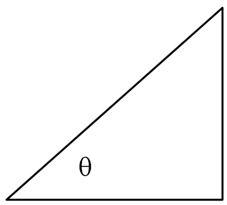
$$\csc\theta = \frac{\text{hypotenuse}}{\text{side opposite}}, \qquad \sec\theta = \frac{\text{hypotenuse}}{\text{side adjacent}}, \qquad \cot\theta = \frac{\text{side adjacent}}{\text{side opposite}}$$

- We will use the triangles and these relationships to develop a method called *trig substitution*, in which each of the factors in the integrand is replaced by a trig expression generated from the appropriate triangle.
- For each expression in the examples above, there were two possible triangles. In each case, we will choose the triangle for which either **sine, tangent, or secant** gives us a fraction with *x* as the numerator and the constant (2 in the above examples) as the denominator.
- These are the three trig functions that are customarily used for trig substitutions.

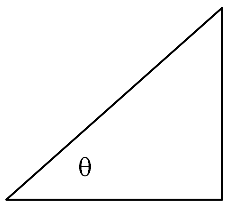
For the following exercise: if θ is the lower angle in the triangle, look at the two corresponding triangles in the previous exercises.

- ✓ Choose the triangle for which either **sine, tangent, or secant** gives us a fraction with ***x* as the numerator and the constant as the denominator**
- ✓ Write the trig substitution that the triangle generates

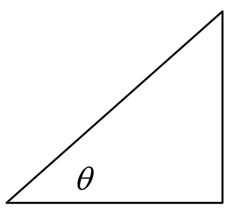
Exercise 4: If the expression is $\sqrt{x^2+4}$ (Exercise 1), the triangle is:



Exercise 5: If the expression is $\sqrt{x^2-4}$ (Exercise 2), the triangle is:



Exercise 6: If the expression is $\sqrt{4-x^2}$ (Exercise 3), the triangle is:



PRACTICE EXERCISES

For the following exercises, draw a right triangle consistent with each expression.

a. Generate an initial trig substitution using the following trig functions: sine, tangent, or secant.

- Remember that the trig function you select should equal a ratio of $\frac{x}{\text{constant}}$.

b. Solve the equation for *x*.

c. Find d*x*.

d. Generate another trig substitution that involves the ratio of $\frac{\text{the given radical}}{\text{constant}}$.

- Note: You can now use cosine, where appropriate.

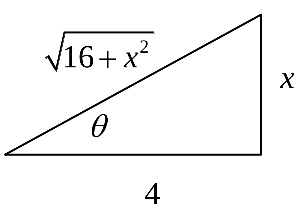
7. $\sqrt{16+x^2}$

8. $\sqrt{x^2-81}$

9. $\sqrt{9-x^2}$

Solutions:

7.



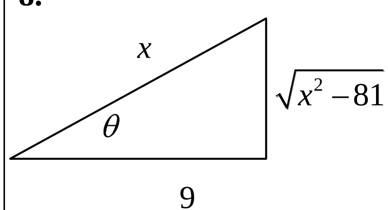
a. $\tan\theta = \frac{x}{4}$

b. $x = 4\tan\theta$

c. $dx = 4\sec^2\theta\, d\theta$

d. $\frac{\sqrt{16+x^2}}{4} = \sec\theta$

8.



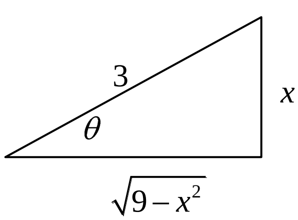
a. $\sec\theta = \frac{x}{9}$

b. $x = 9\sec\theta$

c. $dx = 9\sec\theta\tan\theta\, d\theta$

d. $\frac{\sqrt{81-x^2}}{9} = \tan\theta$

9.



a. $\sin\theta = \frac{x}{3}$

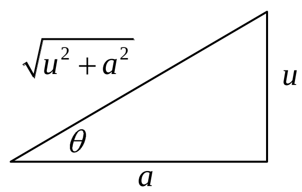
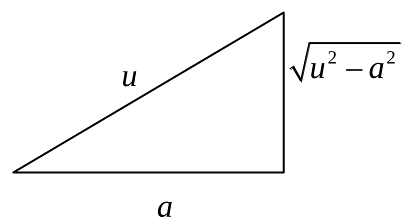
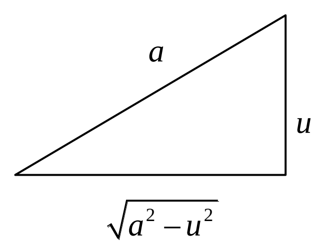
b. $x = 3\sin\theta$

c. $dx = 3\cos\theta\, d\theta$

d. $\frac{\sqrt{9-x^2}}{3} = \cos\theta$

Summary of Expressions, Triangles and Substitutions.

Since we need to represent the most general situations, we will let *u* represent any function of *x*, and *a* is a constant.

Expression	**Triangle**	**Substitution**
$u^2 + a^2$		$\tan\theta = \frac{u}{a}$ or $u = a\tan\theta$
$u^2 - a^2$		$\sec\theta = \frac{u}{a}$ or $u = a\sec\theta$
$a^2 - u^2$		$\sin\theta = \frac{u}{a}$ or $u = a\sin\theta$

Using these triangles and trig relationships, we can take an integral involving one of the three expressions and turn it into a trig integral.